UA Putnam Handout: Block Matrices

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1 Motivation

Question 1. *Show the determinant of an upper triangular matrix is the product of the diagonal entries.*

Question 2. *Given a $m \times n$ matrix A and a $n \times m$ matrix B, show*

$$det(I_n + AB) = det(I_m + BA).$$

2 Big Theorem

The second question seems surprising. How do you relate two non square matrices which don't have determinants to an identity about determinants? As we will see, the answer comes by evaluating the determinant of the block matrix

$$\begin{pmatrix} I_m & -A \\ B & I_n \end{pmatrix}.$$

First, we define what we mean as a block matrix.

Definition 3. A block matrix is a normal matrix which we delineate into blocks. This allows us to think of bigger matrices as a made up of smaller matrices.

For example, we can think of the 4×4 identity matrix as two 2×2 identity matrices and two 2×2 zero matrices:

$$I_4 = \begin{pmatrix} I_2 & O_2 \\ O_2 & I_2 \end{pmatrix}.$$

Now, we will build up some determinant identities.

Lemma 4. *Let* $\Gamma = \begin{pmatrix} I_m & O_{m\times n} \\ O_{n\times m} & A \end{pmatrix}$ *or* $\begin{pmatrix} A & O_{n\times m} \\ O_{m\times n} & I_n \end{pmatrix}$ *where A is an $n \times n$ matrix. Then,* $det(\Gamma) = det(A)$.

Proof. Using Laplace expansion of the m rows which only contain a single 1, we get the result. □

Lemma 5. *Let* $\Gamma = \begin{pmatrix} A & B \\ O_{m\times n} & D \end{pmatrix}$ *or* $\begin{pmatrix} A & O_{n\times m} \\ C & D \end{pmatrix}$ *where* $A : n \times n$, $B : n \times m$, $C : m \times n$, *and* $D : m \times m$. *Then,* $det(\Gamma) = det(A)det(D)$.

Proof. We will show the result for the first case. The second is left as an exercise. Note

$$\Gamma = \begin{pmatrix} A & B \\ O_{m\times n} & D \end{pmatrix} = \begin{pmatrix} I_n & B \\ O_{m\times n} & D \end{pmatrix} \begin{pmatrix} A & O_{n\times m} \\ O_{m\times n} & I_m \end{pmatrix}$$
$$= \begin{pmatrix} I_n & O_{n\times m} \\ O_{m\times n} & D \end{pmatrix} \begin{pmatrix} I_n & B \\ O_{m\times n} & I_m \end{pmatrix} \begin{pmatrix} A & O_{n\times m} \\ O_{m\times n} & I_m \end{pmatrix}.$$

The center matrix is in upper triangular form while the outer matrices are in the form of Lemma 4. Hence, $\det(\Gamma) = \det(D) \cdot 1 \cdot \det(A)$ as desired. □

Theorem 6. *Let* $\Gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ *where* A, B, C, D *have the same sizes as before.*

1. *If* A *is invertible, then*

$$det(\Gamma) = det(A)det(D - CA^{-1}B).$$

2. *If* D *is invertible, then*

$$det(\Gamma) = det(D)det(A - BA^{-1}C).$$

Note how this is similar to the determinant of a 2×2.

Proof. Again, we do the first case and leave the second as an exercise. Suppose A is invertible. Note

$$\Gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I_n & B \\ CA^{-1} & D \end{pmatrix} \begin{pmatrix} A & O_{n\times m} \\ O_{m\times n} & I_m \end{pmatrix}$$
$$= \begin{pmatrix} I_n & O_{n\times m} \\ CA^{-1} & I_m \end{pmatrix} \begin{pmatrix} I_n & B \\ O_{m\times n} & D - CA^{-1}B \end{pmatrix} \begin{pmatrix} A & O_{n\times m} \\ O_{m\times n} & I_m \end{pmatrix}.$$

Hence, the result follows. □

3 Problems

1. (Sylvester's theorem) Given a $m \times n$ matrix A and a $n \times m$ matrix B, show

$$\det(I_n + AB) = \det(I_m + BA).$$

2. (1940 Putnam B6) For integers $a_1, \dots, a_n$ and k, show the determinant of

$$D = \begin{pmatrix} a_1^2 + k & a_1 a_2 & \cdots & a_1 a_n \\ a_2 a_1 & a_2^2 + k & \cdots & a_2 a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_n a_1 & a_n a_2 & \cdots & a_n^2 + k \end{pmatrix}$$

is an integer divisible by k^{n-1} and find $\frac{\det(D)}{k^{n-1}}$.

3. (2018 Putnam B2) Let $S_1, S_2, \dots, S_{2^n-1}$ be the nonempty subsets of $\{1, 2, \dots, n\}$ in some order, and let M be the $(2^n - 1) \times (2^n - 1)$ matrix whose (i, j) entry is

$$m_{ij} = \begin{cases} 0 & \text{if } S_i \cap S_j = \emptyset, \\ 1 & \text{otherwise.} \end{cases}$$

Calculate the determinant of M.

4. Suppose A, B, C, D are $n \times n$ matrix such that AB^t and CD^t are symmetric matrix and $AD^t - BC^t$ is unit matrix. Show $A^t D - C^t B$ is unit matrix.

5. Let X, Y be 7×7 matrices with complex coefficients. If $X^3 + Y^3 = O_7$ and $YX - X^2 Y^2 = I_7$, prove that $XY - Y^2 X^2 = I_7$.