Lecture Handout Complex Numbers

Alabama ARML

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1 Lecture Notes

1.1 Basics Review

Complex Numbers are those of the form $z = a + bi$ where $a, b \in \mathbb{R}$ and $i^2 = -1$. We add complex numbers component-wise and multiply complex numbers using the distributive property:

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

We have two more unitary operations for complex numbers.

Conjugation:

$$\overline{z} = \overline{a + bi} = a - bi$$

Modulus/Absolute Value :

$$|z| = |a + bi| = \sqrt{a^2 + b^2}$$

We can also express complex numbers in polar form

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

Then, multiplication, conjugation, and absolute value take on a different look:

$$z_1 \cdot z_2 = r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)) = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

$$|z| = r$$

$$\overline{z} = re^{-i\theta} = \frac{|z|^2}{z}$$

1.2 Algebra

1.2.1 Polynomials

There are a few expressions we can deal with using complex numbers.

$$Re(z) = a = \frac{z + \overline{z}}{2}$$

$$Im(z) = b = \frac{z - \overline{z}}{2i}$$

$$a^2 + b^2 = (a + bi)(a - bi) = z\overline{z} = |z|^2$$

$$a^2 - b^2 = Re((a + bi)^2)$$

$$2ab = Im((a + bi)^2)$$

$$1 + z + z^2 + ... + z^n = \frac{z^{n+1} - 1}{z - 1}$$

Notice the conjugation operation distributes over addition, multiplication, and exponentiation while the absolute value only distributes over multiplication and exponentiation.

$$\overline{z_1 + z_2 + ... + z_n} = \overline{z_1} + \overline{z_2} + ... + \overline{z_n}$$

$$\overline{z_1 \cdot z_2 \cdot ... \cdot z_n} = \overline{z_1} \cdot \overline{z_2} \cdot ... \cdot \overline{z_n}$$

$$\overline{z^k} = (\overline{z})^k$$

$$|z_1 \cdot z_2 \cdot ... \cdot z_n| = |z_1| \cdot |z_2| \cdot ... \cdot |z_n|$$

$$|z^k| = |z|^k$$

where $n, k \in \mathbb{Z}^+$. These can be easily shown by inspecting components and induction. With this we can lead into a theorem.

Theorem 1. *(Conjugate Root Theorem) For every polynomial $p(x) \in \mathbb{R}[x]$, if z is a root of $p(x)$ then $\overline{z}$ is also a root to $p(x)$.*

This implies that every polynomial with real coefficients has an even number of non-real roots. Another theorem about roots is:

Theorem 2. *(Fundamental Theorem of Algebra) Given a polynomial $p(x) \in \mathbb{C}[x]$ with degree n, $p(x)$ has exactly n complex roots.*

Combining these theorems implies every real odd degree polynomial has at least one real root.

1.2.2 Inequalities

There are not many useful inequalities using complex numbers. However, to investigate roots of a polynomial, sometimes inequalities provide the best answer.

Theorem 3. *(Trivial Inequality/ Triangle Inequality) Let z, z_1, and z_2 be complex numbers. Then,*
(1) $z\overline{z} = |z|^2 \geq 0$
(2) $|z_1| + |z_2| \geq |z_1 + z_2|$
(3) $|z_1| - |z_2| \leq |z_1 - z_2| \leq |z_1| + |z_2|$

1.3 Trigonometry

1.3.1 DeMoivre's Theorem

An infamous pre-calculus theorem.

Theorem 4. *(DeMoivre's Theorem) Let $z = r(\cos\theta + i \sin\theta)$. Then, $z^n = r^n(\cos n\theta + i \sin n\theta)$.*

Recall the $\sin\theta$ and $\cos\theta$ are both 2π periodic. Thus, $z^n = z$ if an only if there exists a positive integer k such that $n\theta = \theta + 2\pi k$. We can conveniently notate this by $n\theta \equiv \theta \pmod{2\pi}$. When θ is a rational multiple of 2π, we can simply use the standard modular arithmetic as 2π divides out of the equation.

1.3.2 Trig Functions

From the polar form of a complex number,

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

1.3.3 Roots of Unity

Theorem 5. *Let* $\omega = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$*. Then,*

(1) $z^n - 1 = (z-1)(z-\omega)(z-\omega^2)...(z-\omega^{n-1})$

(2) $1 + z + z^2 + ... + z^{n-1} = (z-\omega)(z-\omega^2)...(z-\omega^{n-1})$

Notice the solutions of $z^n - r^n = 0$ are scaled roots of unity (ie. r, $r\omega$, ..., $r\omega^{n-1}$).

1.4 Geometry

Complex numbers can be visualized in the Argand or Complex Plane. Complex numbers may be superimposed on a configuration and can help deal with translations, rotations, reflections, and much more.

1.4.1 Translation by Addition

Complex numbers add like vectors.

Theorem 6. *Let point* A *be at complex number* a*,* B *at* b*,* C *at* c*, and* D *at* d*. Then,*

(1) The midpoint $M = m$ *of* AB *is* $m = \frac{a+b}{2}$

(2) A parameterization of the line AB *is* $z(t) = a + t(b-a)$ *for* $t \in \mathbb{R}$.

(3) The centroid $G = g$ *of triangle* ABC *is* $g = \frac{a+b+c}{3}$

(4) Quadrilateral $ABCD$ *is a parallelogram if and only if* $a + c = b + d$

We will denote the premise further by $A = a$, $B = b$, ... etc.

1.4.2 Rotation and Scaling by Multiplication

Instead of using rotation matrices, complex numbers rotate and scale by simple multiplication.

Theorem 7. *(Rotation) Let* z' *be the rotation by angle* α *counterclockwise and scale by a factor of* β *of* $z = re^{i\theta}$ *about the origin* 0*. Then,*

$$z' = \beta e^{i\alpha} z = \beta r e^{i(\theta+\alpha)}$$

1.4.3 Parallel and Perpendicular

It is useful to know the special cases of rotation: Parallel requires 0 rotation while perpendicular requires $\frac{\pi}{2}$.

Theorem 8. *Consider the segments* AB *and* CD *where* $A = a$*,* $B = b$*,* $C = c$*, and* $D = d$*. Let* $p = \frac{b-a}{d-c}$*. Then,*

(1) p *is real (or* $p = \overline{p}$*) if and only if* AB *is parallel to* CD

(2) p *is imaginary (or* $p = -\overline{p}$*) if and only if* AB *is perpendicular to* CD

1.4.4 Reflection by Conjugation

Notice that $\overline{z}$ is a reflection of z about the real axis. We can use this fact to more generally reflection complex numbers about arbitrary line segments.

Theorem 9. *(Reflection) Let* z' *be the reflection of* z *about the line* AB*. Then,*

$$\overline{\left(\frac{z-a}{b-a}\right)} = \frac{z'-a}{b-a}$$

1.4.5 Example: The Euler Line

Theorem 10. *(The Euler Line) Without loss of generality, suppose triangle* ABC *has circumcenter* O *at* 0 *and* BC *parallel to the real axis. Then,*

(1) Centroid G *is at* $g = \frac{a+b+c}{3}$

(2) Orthocenter H *is at* $h = a + b + c$

(3) O*,* H*,* G *line of the line* $z(t) = (a + b + c)t$

(4) $\frac{HG}{GO} = 2$

2 Problems

2.1 Algebra

1. (2019 Alabama Statewide Comp. Q 6) Write $\sqrt{16+30i}$ in form $a+bi$ where $a,b \geq 0$.

2. (2020 Alabama Statewide Comp. Q 27) Let $(x,y) \in \mathbb{C}^2$ with $y \neq 0$. If $x^4 + 2x^2y^2 + 9y^2 = 0$, find all possible values of $\left|\frac{x}{y}\right|$.

3. (2019 AMC 12A Q 14) For a certain complex number c, the polynomial

$$P(x) = (x^2 - 2x + 2)(x^2 - cx + 4)(x^2 - 4x + 8)$$

has exactly 4 distinct roots. What is $|c|$?

(A) 2 **(B)** $\sqrt{6}$ **(C)** $2\sqrt{2}$ **(D)** 3 **(E)** $\sqrt{10}$

4. (2020 AMC 12A Q 22) Let (a_n) and (b_n) be the sequences of real numbers such that

$$(2+i)^n = a_n + b_n i$$

for all integers $n \geq 0$, where $i = \sqrt{-1}$. What is

$$\sum_{n=0}^{\infty} \frac{a_n b_n}{7^n}\ ?$$

(A) $\frac{3}{8}$ **(B)** $\frac{7}{16}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{9}{16}$ **(E)** $\frac{4}{7}$

5. (2018 AMC 12A Q 22) The solutions to the equations $z^2 = 4+4\sqrt{15}i$ and $z^2 = 2+2\sqrt{3}i$, where $i = \sqrt{-1}$, form the vertices of a parallelogram in the complex plane. The area of this parallelogram can be written in the form $p\sqrt{q} - r\sqrt{s}$, where p, q, r, and s are positive integers and neither q nor s is divisible by the square of any prime number. What is $p+q+r+s$?

(A) 20 **(B)** 21 **(C)** 22 **(D)** 23 **(E)** 24

6. (2018 ARML Team Q 5) The absolute values of the 5 complex roots of $z^5 + 5z^2 + 53 = 0$ all lie between the positive integers a and b, where $a < b$ and $b - a$ is minimal. Compute the ordered pair (a,b).

7. Let m and n be positive integers that can both be written as the sum of two perfect squares. Prove the mn also has this property. For example, $m = 17$, $n = 13$, and $mn = 221$ all have this property.

8. Suppose the a, b, u, and v are all real numbers such that $av - bu = 1$. Prove that

$$a^2 + b^2 + u^2 + v^2 + au + bv \geq \sqrt{3}$$

2.2 Trigonometry

1. (2018 Alabama Statewide Comp. Q 33) Find the largest positive integer n on $[0,100]$ such that $(1+i\sqrt{3})^n$ is a real number.

2. (2019 AMC 12A Q 21) Let

$$z = \frac{1+i}{\sqrt{2}}.$$

What is

$$\left(z^{1^2} + z^{2^2} + z^{3^2} + \cdots + z^{12^2}\right) \cdot \left(\frac{1}{z^{1^2}} + \frac{1}{z^{2^2}} + \frac{1}{z^{3^2}} + \cdots + \frac{1}{z^{12^2}}\right)?$$

(A) 18 **(B)** $72 - 36\sqrt{2}$ **(C)** 36 **(D)** 72 **(E)** $72 + 36\sqrt{2}$

3. (2018 AMC 12B Q 16) The solutions to the equation $(z+6)^8 = 81$ are connected in the complex plane to form a convex regular polygon, three of whose vertices are labeled A, B, and C. What is the least possible area of $\triangle ABC$?

   (A) $\frac{1}{6}\sqrt{6}$ **(B)** $\frac{3}{2}\sqrt{2} - \frac{3}{2}$ **(C)** $2\sqrt{3} - 3\sqrt{2}$ **(D)** $\frac{1}{2}\sqrt{2}$ **(E)** $\sqrt{3} - 1$

4. (2020 AMC 12A Q 15) In the complex plane, let A be the set of solutions to $z^3 - 8 = 0$ and let B be the set of solutions to $z^3 - 8z^2 - 8z + 64 = 0$. What is the greatest distance between a point of A and a point of B?

   (A) $2\sqrt{3}$ **(B)** 6 **(C)** 9 **(D)** $2\sqrt{21}$ **(E)** $9 + \sqrt{3}$

5. (2020 AMC 12B Q 17) How many polynomials of the form $x^5 + ax^4 + bx^3 + cx^2 + dx + 2020$, where a, b, c, and d are real numbers, have the property that whenever r is a root, so is $\dfrac{-1 + i\sqrt{3}}{2} \cdot r$? (Note that $i = \sqrt{-1}$)

   (A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

6. (2019 ARML Team Q 6) A complex number z is selected uniformly at random such that $|z| = 1$. Compute the probability the z and z^{2019} both lie in Quadrant II of the complex plane.

7. Find a closed-form expression for each of the quantites:

$$\sum_{k=0}^{n} \sin(k)$$

   and

$$\prod_{k=1}^{n-1} \sin\left(\frac{k\pi}{n}\right)$$

8. Factor $x^5 + x + 1$

9. Show that $\cos(\frac{\pi}{100})$ is irrational.

2.3 Geometry

1. (2018 AMC 12B Q 17) How many nonzero complex numbers z have the property that $0, z$, and z^3, when represented by points in the complex plane, are the three distinct vertices of an equilateral triangle?

   (A) 0 **(B)** 1 **(C)** 2 **(D)** 4 **(E)** infinitely many

2. (2019 AMC 12B Q 24) Let $\omega = -\frac{1}{2} + \frac{1}{2}i\sqrt{3}$. Let S denote all points in the complex plane of the form $a + b\omega + c\omega^2$, where $0 \le a \le 1, 0 \le b \le 1$, and $0 \le c \le 1$. What is the area of S?

   (A) $\frac{1}{2}\sqrt{3}$ **(B)** $\frac{3}{4}\sqrt{3}$ **(C)** $\frac{3}{2}\sqrt{3}$ **(D)** $\frac{1}{2}\pi\sqrt{3}$ **(E)** π

3. (2020 AMC 12B Q 23) How many integers $n \ge 2$ are there such that whenever $z_1, z_2, ..., z_n$ are complex numbers such that

$$|z_1| = |z_2| = ... = |z_n| = 1 \text{ and } z_1 + z_2 + ... + z_n = 0,$$

   then the numbers $z_1, z_2, ..., z_n$ are equally spaced on the unit circle in the complex plane?

   (A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

4. (2020 AIME I Q 8) A bug walks all day and sleeps all night. On the first day, it starts at point O, faces east, and walks a distance of 5 units due east. Each night the bug rotates 60° counterclockwise. Each day it walks in this new direction half as far as it walked the previous day. The bug gets arbitrarily close to the point P. Then $OP^2 = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

5. Show that given any quadrilateral, the midpoints of its sides form a parallelogram.

6. Prove the law of cosines.
7. (1994 AIME Q 8) The points $(0,0),(a,11)$, and $(b,37)$ are vertices of an equilateral triangle. Find the value ab
8. Consider a regular n-gon which is inscribed in a circle with radius 1. Compute the product of the lengths of all $\frac{n(n-3)}{2}$ diagonals and n sides of this polygon.
9. Given any $\triangle ABC$, construct point E_A such that $\triangle BE_AC$ is an equilateral triangle, with E_A being on the opposite side of BC as A. Let N_A be the centroid of $\triangle BE_AC$. Similarly define N_B and N_C. Prove that $\triangle N_AN_BN_C$ is equilateral.
10. (2015 USAMO Q 2) Quadrilateral $APBQ$ is inscribed in circle ω with $\angle P = \angle Q = 90^\circ$ and $AP = AQ < BP$. Let X be a variable point on segment $\overline{PQ}$. Line AX meets ω again at S (other than A). Point T lies on arc AQB of ω such that $\overline{XT}$ is perpendicular to $\overline{AX}$. Let M denote the midpoint of chord $\overline{ST}$. As X varies on segment $\overline{PQ}$, show that M moves along a circle.