Alabama ARML Handout: Burnside's Lemma

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1 Motivation

Question 1. *Find the number of ways to color each corner of a square red or blue.*

Question 2. *Two colorings are considered the same if one is a rotation of the other. Find the number of different ways to color each corner of a square red or blue.*

2 Big Theorem

Let's abstract what is going on with this example. Let X be the set we are trying to count and G be its underlying set of symmetries.

> In the example above, X is the set of all possible colorings of the square and G are the rotations by $0°, 90°, 180°$, and $270°$.

We denote these rotations by $R_0, R_{90}, R_{180}, R_{270}$. Why do we need the zero rotation? It is important that G forms a *group*.

Definition. We say G is a group if it has a binary operation $\star$ that satisfies

1. $x \star y \in G$ for all $x, y \in G$ (closed),
2. $x \star (y \star z) = (x \star y) \star z$ for all $x, y, z \in G$ (associativity),
3. there exists an identity $e \in G$ such that $x \star e = e \star x = x$ for all $x \in G$ (identity),
4. each $x \in G$ has an inverse $x^{-1} \in G$ so that $x \star x^{-1} = x^{-1} \star x = e$ (inverses).

This is a very technical definition but the key take-aways are that combining the symmetries give other symmetries and we can always undo any symmetry by its inverse. This way we have all the symmetries. Define for each $g \in G$, X^g is the things in X which remain fixed by g ie.

$$X^g = \{x \in X \ : \ g(x) = x\}$$

where $g(x) \in X$ is the element where g is applied to x. We can now state Burnside's lemma.

Theorem 3. *(Burnside's Lemma) Let X be the set we want to count and G its underlying group of symmetries. If we consider two elements of X the same when we can obtain one by applying a symmetry in G to the other, then the number of different elements in X is given by*

$$\frac{1}{|G|}\sum_{g\in G}|X^g|.$$

Proof. The proof requires some standard group theory arguments. However, the argument boils down to double counting the set

$$\{(g,x)\ :\ g(x)=x\}$$

and interchanging a double sum. □

3 Examples

We will work two examples in this section. First, we will solve the motivating question.

Solution to Question 2 above As we said about, let X be the set of all possible colorings of the square and $G = \{R_0, R_{90}, R_{180}, R_{270}\}$. Convince yourself that G does form a group using the definition above. Then, the number of different colorings is

$$\frac{2^4+2^1+2^2+2^1}{4} = \boxed{6}.$$

Now, for a new question:

Question 4. *There are ten chairs at a circular table and ten people sit down. If two seating arrangements are considered the same if one is a rotation of the other, then how many different seating arrangements are there?*

Solution By a a standard overcounting argument, there are $\boxed{9!}$ arrangements. This agrees with Burnside's lemma! Let X be all the possible seating arrangements and $G = \{R_0, R_{36}, R_{72}, \ldots, R_{324}\}$. However, the only symmetry which fixes an arrangement is the identity R_0 (and it fixes every arrangement). Hence, we have there are

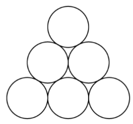
$$\frac{10!+0+\cdots+0}{10} = \boxed{9!}.$$

4 Problems

Here are some problems which you can use Burnside's lemma to solve.

1. Find the number of distinct patterns that can be formed from shading three of the nine squares in a 3×3 grid of unit squares. Two patterns are not considered distinct if one can be obtained from the other through a series of flips and/or rotations.

2. In the figure below, 3 of the 6 disks are to be painted blue, 2 are to be painted red, and 1 is to be painted green. Two paintings that can be obtained from one another by a rotation or a reflection of the entire figure are considered the same. How many different paintings are possible? (2017 AMC 12B # 13)



(A) 6 **(B)** 8 **(C)** 9 **(D)** 12 **(E)** 15

3. Each corner of a cube is color red, blue or green. Find the number of distinct cubes that can be formed. Two cubes are not distinct if one can be rotated to become the other.

4. Each square of a 2×2 grid of unit squares is colored yellow, blue, red, or green. Find the number of distinct grids that can be obtained. Two patterns are indistinguishable if they can be reflected or rotated to become the other.

5. Wally's Key Company makes and sells two types of keys. Mr. Porter buys a total of 12 keys from Wally's. Determine the number of arrangements of Mr. Porter's 12 new keys on his keychain. (Where rotations are considered the same and any two keys of the same type are identical) (Thomas Mildorf Mock AIME I # 14)

6. Edward places 7 beads on a bracelet chain, then ties the ends together. The beads are identical except for color. If 3 of the beads are red, 2 are green, and 2 are blue, how many distinct bracelets are possible? Two bracelets are identical if one can be rotated or flipped to produce the other.

7. Two quadrilaterals are considered the same if one can be obtained from the other by a rotation and a translation. How many different convex cyclic quadrilaterals are there with integer sides and perimeter equal to 32? (2010 AMC 12A # 25)

   (A) 560 **(B)** 564 **(C)** 568 **(D)** 1498 **(E)** 2255

8. David builds a circular table; he then carves one or more positive integers into the table at points equally spaced around its circumference. He considers two tables to be the same if one can be rotated so that it has the same numbers in the same positions as the other. For example, a table with the numbers 8, 4, 5 (in clockwise order) is considered the same as a

table with the numbers 4, 5, 8 (in clockwise order), but both tables are different from a table with the numbers 8, 5, 4 (in clockwise order). Given that the numbers he carves sum to 17, compute the number of different tables he can make. (2023 ARML Team # 6)

9. 27 unit cubes - 25 of which are colored black and 2 of which are colored white - are assembled to form a $3 \times 3 \times 3$ cube. How many distinguishable cubes can be formed? (Two cubes are indistinguishable if one of them can be rotated to appear identical to the other.)

10. Determine the number of unordered triples of nonnegative integers (x, y, z) such that $x + y + z = 2023$.

11. Determine the number of unordered triples of positive integers (m, n, p) such that $mnp = 72$.